# RISOLUZIONE ESERCIZIO SUGLI INTEGRALI DOPPI

In questo documento (liberamente scaricabile) troverai la risoluzione di un esercizio svolto da me (Nicola Tarantino) sugli integrali doppi. Per risolvere questo esercizio, data la simmetria del dominio, ho ritenuto opportuno effettuare il cambiamento di variabili in coordinate polari.

Calcolare $\iint_E \min\left\{1, \frac{1}{\sqrt{x^2+y^2}}\right\} dxdy$,
dove $E=\{(x,y)\in\mathbf{R}^2 | x\geq 0,\ y\geq 0,\ x^2+y^2-2x\leq 0\}$.

Conviene innanzitutto passare a coordinate polari. Posti $x=\rho\cos\theta$ e $y=\rho\sin\theta$, si ha che l'insieme E viene trasformato nell'insieme $\hat{E}=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{2},\ 0\leq\rho\leq 2\cos\theta\right\}$ e quindi

$$\iint_E \min\left\{1, \frac{1}{\sqrt{x^2+y^2}}\right\} dxdy = \iint_{\hat{E}} \rho\min\left\{1, \frac{1}{\rho}\right\} d\rho d\theta.$$

Osserviamo che $\min\left\{1, \frac{1}{\rho}\right\} = \begin{cases} 1 & \text{se } \rho\leq 1 \\ \frac{1}{\rho} & \text{se } \rho\geq 1 \end{cases}$.

Da ciò si evince che

$$\hat{E}=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{2},\ 0\leq\rho\leq\min\{1,2\cos\theta\}\right\}\cup$$
$$\cup\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{2},\ 1\leq\rho\leq 2\cos\theta\right\}.$$

$$\min\{1,2\cos\theta\} = \begin{cases} 1 & \text{se } 1\leq 2\cos\theta \\ 2\cos\theta & \text{se } 2\cos\theta\leq 1 \end{cases} = \begin{cases} 1 & \text{se } 0\leq\theta\leq\frac{\pi}{3} \\ 2\cos\theta & \text{se } \frac{\pi}{3}\leq\theta\leq\frac{\pi}{2} \end{cases}$$

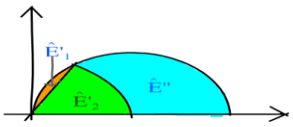
Di conseguenza

$$\hat{E}=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{3},\ 0\leq\rho\leq 1\right\}\cup$$
$$\cup\left\{(\rho,\theta)\in\mathbf{R}^2 |\ \frac{\pi}{3}\leq\theta\leq\frac{\pi}{2},\ 0\leq\rho\leq 2\cos\theta\right\}\cup$$
$$\cup\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{3},\ 1\leq\rho\leq 2\cos\theta\right\}$$

se $\frac{\pi}{3}\leq\theta\leq\frac{\pi}{2},\ 2\cos\theta\leq 1$

Denotati con:

$$\hat{E}'_1=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{3},\ 0\leq\rho\leq 1\right\},$$
$$\hat{E}'_2=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ \frac{\pi}{3}\leq\theta\leq\frac{\pi}{2},\ 0\leq\rho\leq 2\cos\theta\right\},$$
$$\hat{E}''=\left\{(\rho,\theta)\in\mathbf{R}^2 |\ 0\leq\theta\leq\frac{\pi}{3},\ 1\leq\rho\leq 2\cos\theta\right\},$$




risulta che

$$\iint_{\hat{E}} \rho\min\left\{1, \frac{1}{\rho}\right\} d\rho d\theta = \iint_{\hat{E}'_1} \rho d\rho d\theta + \iint_{\hat{E}'_2} \rho d\rho d\theta + \iint_{\hat{E}''} d\rho d\theta =$$
$$= \int_0^{\frac{\pi}{3}} d\theta \int_0^1 \rho d\rho + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} \rho d\rho\right) d\theta + \int_0^{\frac{\pi}{3}} \left(\int_1^{2\cos\theta} d\rho\right) d\theta =$$
$$= \frac{\pi}{6} + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta + \int_0^{\frac{\pi}{3}} (2\cos\theta - 1) d\theta =$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$= \frac{\pi}{6} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta + [2\sin\theta - \theta]_0^{\frac{\pi}{3}} =$$
$$= \frac{\pi}{6} + \left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \sqrt{3} - \frac{\pi}{3} = \frac{\pi}{6} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{3} + \sqrt{3} - \frac{\pi}{3} =$$
$$= \frac{3}{4}\sqrt{3} - \frac{\pi}{3}.$$